Lifting, Duality, Voronoi and Delaunay

Our goal is to identify the Voronoi and Delaunay diagrams in $\mathbb{R}^d$ as projection of polyhedra in $\mathbb{R}^{d+1}$. Though, the presentation is for arbitrary d, you may want to replace $d = 2$ everywhere as you read the first time. For completeness, we present some basic concepts about convex polyhedra and polytopes; in some cases without proof: they are not very complicated but also non-trivial; for $d = 2, 3$ you may rely on your intuition. We begin by making precise the concept of a *complex*, of which the diagrams of Voronoi and Delaunay are examples.

Complexes and Simplicial Complexes. A *complex* $\mathcal{C}$ is a collection of (relatively closed) *faces or cells* such that if $f, g \in \mathcal{C}$ then $f \cap g$ is either empty or is in $\mathcal{C}$. If $f, g \in \mathcal{C}$ and $g \subset f$, we say that g is a face of f. We also say that f is *incident* to g. The *underlying space* of $\mathcal{C}$, denoted $|\mathcal{C}|$ is $\bigcup\{f : f \in \mathcal{C}\}$,[1] the union of all the faces in the complex. We are interested in the special case in which each face is a convex polyhedron (defined below). A k-simplex $0 \leq k \leq d$, in $\mathbb{R}^d$ is the convex hull of a set $k+1$ affinely independent points[2]. If each of the cells in a complex $\mathcal{C}$ is a simplex, we say that the complex is *simplicial.*

Example. A familiar example is a triangulation of a set of points in the plane. This is a simplicial complex and its underlying space is the convex hull of the point set. In contrast, the trapezoidation of a set of segments in the plane is not a complex. The Voronoi diagram is a complex that is not simplicial.

Polytopes and polyhedra. For a set $X \subseteq \mathbb{R}^d$, its *convex hull* $\text{conv}(X)$ is the smallest convex set containing X. A *convex polytope* in $\mathbb{R}^d$ is the convex hull of a finite point. A *convex polyhedron* is the intersection of a finite set of halfspaces. A polyhedron can be unbounded while a polytope is bounded.

◀ *A convex polytope is a convex polyhedron. A bounded convex polyhedron is a convex polytope.*

Let $\mathcal{P}$ be a polyhedron. A hyperplane h *supports* $\mathcal{P}$ if it intersects $\mathcal{P}$ and a halfspace bounded by h contains $\mathcal{P}$. Thus, h intersect the boundary of $\mathcal{P}$, denoted $\text{bd}(\mathcal{P})$. If h supports $\mathcal{P}$, then $h \cap \mathcal{P}$ is called a *face* of $\mathcal{P}$, and it is said to be supported by h. A face is also a polyhedron. A face of dimension k is called a *k-face.* 0-faces are called *vertices*, 1-faces are called *edges*, $(d-2)$-faces are called *ridges* and $(d-1)$-faces are called *facets.* A face f is also a polyhedron (and a polytope if so is $\mathcal{P}$). A point x in $X \subseteq \mathbb{R}^d$ is *extremal* if $x \notin \text{conv}(X \setminus \{x\})$.

◀ *The extremal points of a polyhedron are exactly its vertices. A polytope is the convex hull of its vertices.*

The faces of a face f of a polyhedron $\mathcal{P}$ are exactly those faces of $\mathcal{P}$ that are contained in $\mathcal{P}$. Thus,

[1] For a collection $\mathcal{F}$ of sets, we write $\bigcup \mathcal{F}$ to mean $\bigcup_{A \in \mathcal{F}} A$. Similarly for $\bigcap$.

[2] The affine space determined by points $\mathbf{p}_0, \ldots, \mathbf{p}_k \in \mathbb{R}^d$, denoted $\text{affine}(\mathbf{p}_0, \ldots, \mathbf{p}_k)$, is the set of all the points of the form $\sum_{j=0}^{k} \lambda_i \mathbf{p}_i$ where $\lambda_i \in \mathbb{R}$. These points are affinely independent if $\sum_{j=0}^{k} \lambda_i \mathbf{p}_i = 0$ iff $\lambda_i = 0$ for each i (and then the affine space is k-dimensional). A k-dimensional affine space is also called a *k-flat.* (We use "iff" as short for "if and only if".)

◀ *The set of all faces of a convex polyhedron is a complex.*

This is called the *boundary complex* of $\mathcal{P}$. $\mathcal{P}$ is *simple* if a k-face is the intersection of exactly $d-k$ facets, for $k = 1, \ldots, d-1$. In particular, a vertex is the intersection of d facets. A polytope $\mathcal{P}$ is *simplicial* if each k-face is a k-simplex with its $k+1$ vertices in P.
Example. In $\mathbb{R}^3$, a polyhedron is simple if each vertex has exactly three incident facets (and also three incident edges). A polytope is simplicial if each facet is a triangle.

Lifting and Projection. Let $\pi_d : \mathbb{R}^{d+1} \to \mathbb{R}^d$ be the *vertical projection*

$$\pi_d(x_0, \ldots, x_{d-1}, x_d) = (x_0, \ldots, x_{d-1}).$$

We use a symbol with a hat to denote an object in $\mathbb{R}^{d+1}$ and the corresponding symbol without hat to denote its projection under π_d. Thus, for example, we write $\hat{\mathbf{x}} = (\mathbf{x}, x_d) \in \mathbb{R}^{d+1}$ where $\mathbf{x} = \pi_d(\hat{\mathbf{x}}) \in \mathbb{R}^d$. For $\mathbf{x} \in \mathbb{R}^d$, $\pi_d^{-1}(\mathbf{x})$ is the vertical line of points that project onto $\mathbf{x}$. Of all these possible liftings of $\mathbf{x}$, an important one is $\lambda_d : \mathbb{R}^d \to \mathbb{R}^{d+1}$ be the *lifting to the paraboloid*

$$\Pi_{d+1} = \{\hat{\mathbf{x}} = (\mathbf{x}, x_d) \in \mathbb{R}^{d+1} : x_d = \|\mathbf{x}\|^2\}$$

defined by

$$\lambda_d(\mathbf{x}) = (\mathbf{x}, \|\mathbf{x}\|^2) \in \mathbb{R}^{d+1}.$$

Primal and dual spaces. Let the *primal* and *dual* spaces, denoted $\hat{\mathsf{P}}$ and $\hat{\mathsf{D}}$, be copies of $\mathbb{R}^{d+1}$ (it is not really necessary to think of them as different spaces). Let $\mathsf{P} = \pi_d(\hat{\mathsf{P}})$ and $\mathsf{D} = \pi_d(\hat{\mathsf{D}})$. For $\hat{\mathbf{p}} = (\mathbf{p}, p_d) \in \mathbb{R}^{d+1}$, let

$$\mathcal{D}(\hat{\mathbf{p}}) = \{\hat{\mathbf{y}} \in \mathbb{R}^{d+1} : y_d = 2\mathbf{p} \cdot \mathbf{y} - p_d\},$$

where $\cdot$ denotes vector product, which is a non-vertical hyperplane in $\mathbb{R}^{d+1}$. On the other hand, for a non-vertical hyperplane $h_{\hat{\mathbf{m}}} = \{\hat{\mathbf{x}} \in \mathbb{R}^{d+1} : x_d = \mathbf{m} \cdot \mathbf{x} + m_d\}$ in $\mathbb{R}^{d+1}$, let

$$\mathcal{D}(h_{\hat{\mathbf{m}}}) = (\mathbf{m}/2, -m_d),$$

a point in $\mathbb{R}^{d+1}$. This establishes a relation between points in $\hat{\mathsf{P}}$ (resp. $\hat{\mathsf{D}}$) and non-vertical hyperplanes in $\hat{\mathsf{D}}$ (resp. $\hat{\mathsf{P}}$).

Duality is an involution.

◀ $\mathcal{D}(\mathcal{D}(\hat{\mathbf{p}})) = \hat{\mathbf{p}}$ *and* $\mathcal{D}(\mathcal{D}(h_{\hat{\mathbf{m}}})) = h_{\hat{\mathbf{m}}}$.

Duality preserve incidences:

◀ $\hat{\mathbf{p}}$ *is above (resp. on, below)* $h_{\hat{\mathbf{m}}}$ *in* $\hat{\mathsf{P}}$ *iff* $\mathcal{D}(h_{\hat{\mathbf{m}}})$ *is above (resp. on, below)* $\mathcal{D}(\hat{\mathbf{p}})$ *in* $\hat{\mathsf{D}}$.

Proof. It follows from the following trivial sequence of equivalences:

$$\begin{aligned}
\hat{\mathbf{p}} \text{ is above } h_{\hat{\mathbf{m}}} \quad & \text{iff} \quad p_d > \mathbf{m} \cdot \mathbf{p} + m_d \\
& \text{iff} \quad -m_d > 2\mathbf{p} \cdot (\mathbf{m}/2) - p_d \\
& \text{iff} \quad (\mathbf{m}/2, -m_d) \text{ above the hyperplane } y_d = 2\mathbf{p} \cdot \mathbf{y} - p_d \\
& \text{iff} \quad \mathcal{D}(h_{\hat{\mathbf{m}}}) \text{ is above } \mathcal{D}(\hat{\mathbf{p}}).
\end{aligned}$$

The same argument applies for the on and below relations. □

(The constant 2 in the definition of the dual is not very important. We could omit it, but the we would need to work with the paraboloid $\|\mathbf{x}\|^2/2$ so that all the desired relations for Delaunay and Voronoi diagrams hold.)

Lower Hulls and Upper Envelopes. For a non-vertical hyperplane $h_{\hat{\mathbf{m}}}$ in $\mathbb{R}^{d+1}$, let $h^+_{\hat{\mathbf{m}}}$ denote the *upper* (closed) halfspace in $\mathbb{R}^{d+1}$ bounded from below by $h_{\hat{\mathbf{m}}}$. Let $\hat{S}$ be a finite point set in $\hat{\mathsf{P}}$. Let the *lower hull* of $\hat{S}$ be

$$\mathcal{P}(\hat{S}) = \bigcap\{h^+_{\hat{\mathbf{m}}} : \hat{\mathbf{m}} \in \mathbb{R}^{d+1}, \hat{S} \subseteq h^+_{\hat{\mathbf{m}}}\},$$

that is, the intersection of all the upper halfspaces that contain $\hat{S}$, and let $\mathsf{LH}(\hat{S})$ (*lower hull complex*) consist of the faces in the boundary complex of the convex hull of $\hat{S}$ that are supported by non-vertical hyperplanes from below (that is, all except the infinite faces). We assume that $\hat{S}$ is in *general position*, that is, no $k+2$ points lie in a k-flat (k-dimensional affine space), for $k = 0, 1, \ldots, d$. Under this condition, $\mathsf{LH}(\hat{S})$ is simplicial. In $\hat{\mathsf{D}}$, consider the dual hyperplanes $\mathcal{D}(\hat{S}) = \{\mathcal{D}(\hat{\mathbf{p}}) : \hat{\mathbf{p}} \in \hat{S}\}$, and let *upper cell* of $\mathcal{D}(\hat{S})$ be

$$\mathcal{Q}(\mathcal{D}(\hat{S})) = \bigcap\{h^+ : h \in \mathcal{D}(\hat{S}))\},$$

the intersection of the corresponding upper halfspaces Note that:

◀ $\hat{\mathbf{x}} \in \mathcal{P}(\hat{S})$ *iff* $\mathcal{Q}(\mathcal{D}(\hat{S})) \subseteq \mathcal{D}(\hat{\mathbf{x}})^+$.

Proof. It follows from the following trivial sequence of equivalences:

$$\begin{aligned}
\hat{\mathbf{x}} \in \mathcal{P}(\hat{S}) \quad &\text{iff} \quad \forall h_{\hat{\mathbf{m}}}[(\forall \hat{\mathbf{p}} \in \hat{S},\ \hat{\mathbf{p}} \in h^+_{\hat{\mathbf{m}}}) \Rightarrow \hat{\mathbf{x}} \in h^+_{\hat{\mathbf{m}}}] \\
&\text{iff} \quad \forall h_{\hat{\mathbf{m}}}[(\forall \hat{\mathbf{p}} \in \hat{S},\ \mathcal{D}(h_{\hat{\mathbf{m}}}) \in \mathcal{D}(\hat{\mathbf{p}})^+) \Rightarrow \mathcal{D}(h_{\hat{\mathbf{m}}}) \in \mathcal{D}(\hat{\mathbf{x}})^+] \\
&\text{iff} \quad \forall \hat{\mathbf{y}}[(\forall \hat{\mathbf{p}} \in \hat{S},\ \hat{\mathbf{y}} \in \mathcal{D}(\hat{\mathbf{p}})^+) \Rightarrow \hat{\mathbf{y}} \in \mathcal{D}(\hat{\mathbf{x}})^+] \\
&\text{iff} \quad \mathcal{Q}(\mathcal{D}(\hat{S})) \subseteq \mathcal{D}(\hat{\mathbf{x}})^+.
\end{aligned}$$

□

Let $\mathsf{UE}(\mathcal{D}(\hat{S}))$ (*upper envelope complex*) be the boundary complex of $\mathcal{Q}(\mathcal{D}(\hat{S}))$. Let $\hat{V}_{\hat{\mathbf{p}}}$ be the facet of $\mathcal{Q}(\mathcal{D}(\hat{S}))$ supported by $h_{\hat{\mathbf{p}}}$ (which may be empty). The faces in $\mathsf{LH}(\hat{S})$ and in $\mathsf{UE}(\mathcal{D}(\hat{S}))$ are in one-to-one correspondence:

◀ *For $\hat{T} \subseteq \hat{S}$, conv($\hat{T}$) is a k-face in $\mathsf{LH}(\hat{S})$ iff $\bigcap\{\hat{V}_{\hat{\mathbf{p}}} : \hat{\mathbf{p}} \in \hat{T}\}$ is a $(d-k)$-face in $\mathsf{UE}(\mathcal{D}(\hat{S}))$.*

In particular, if $\hat{S}$ is in general position, and so $\mathsf{LH}(\hat{S})$ is simple, then $\mathsf{UE}(\mathcal{D}(\hat{S}))$ is simplicial.

Delaunay and Voronoi Complexes. Let $S \subseteq \mathbb{R}^d$ be a finite set of points, that we call *sites*, in general position (any $k+1$ points are affinely independent, $0 \le k \le d$; any $d+2$ points are not co-spherical). For $T \subseteq S$ of size $d+1$, sphere(T) denotes the circumsphere of T. The *Delaunay complex* $\mathsf{Del}(S)$ consists of all the d-simplices $\triangle = \text{conv}(T)$ such that sphere(T) contains no sites in its interior, and its faces. For a site $\mathbf{p} \in S$, its *Voronoi cell* $V_{\mathbf{p}}$ is the set of points that are no closer to any other other site. The *Voronoi complex* $\mathsf{Vor}(S)$ consists of all the Voronoi cells $V_{\mathbf{p}}$ and its faces. It is not obvious from the definition that $\mathsf{Del}(S)$ and $\mathsf{Vor}(S)$ are complexes. This will follow from the lifting to the boundary complex of a polyhedron in $\mathbb{R}^{d+1}$.

◀ *Let $T \subseteq S$. conv(T) is a k-face in $\mathsf{Del}(S)$ iff $\bigcup_{\mathbf{p} \in T} V_{\mathbf{p}}$ is a $(d-k)$-face in $\mathsf{Vor}(S)$.*

Lifting Spheres and Bisectors. The sphere $s(\mathbf{c}, r)$ in $\mathbb{R}^d$ with center in $\mathbf{c} \in \mathbb{R}^d$ and radius $r \geq 0$ is the set $\{\mathbf{x} \in \mathbb{R}^d : ||x-c||^2 = r^2\}$. Rewriting this as $||\mathbf{x}||^2 = 2\mathbf{c}\cdot\mathbf{x} + (r^2 - ||\mathbf{c}||^2)$, we see that $s(\mathbf{c}, r)$ is the projection under π_d from the intersection in $\mathbb{R}^{d+1}$ between the paraboloid Π_{d+1} and the hyperplane

$$\{\hat{\mathbf{x}} \in \mathbb{R}^{d+1} : x_d = 2\mathbf{c} \cdot \mathbf{x} + (r^2 - ||\mathbf{c}||^2)\}$$

which is precisely $\mathcal{D}((\mathbf{c}, ||\mathbf{c}||^2 - r^2))$. It also follows that the projection of the intersection of any hyperplane and Π_{d+1}, if nonempty, is a sphere in $\mathbb{R}^d$. Also, making $r \to 0$, we obtain that the hyperplane $\mathcal{D}((\mathbf{c}, ||\mathbf{c}||^2)) = \{\hat{\mathbf{x}} \in \mathbb{R}^{d+1} : x_d = 2\mathbf{c} \cdot \mathbf{x} - ||\mathbf{c}||^2\}$ is tangent to Π_{d+1} at $(\mathbf{c}, ||\mathbf{c}||^2)$.

Now consider the bisector between $\mathbf{p}, \mathbf{q} \in \mathbb{R}^d$: $||\mathbf{x} - \mathbf{p}||^2 = ||\mathbf{x} - \mathbf{q}||^2$. Expanding and cancelling the terms $||\mathbf{x}||^2$, we obtain $2\mathbf{p}\cdot\mathbf{x} - ||\mathbf{p}||^2 = 2\mathbf{q}\cdot\mathbf{x} - ||\mathbf{q}||^2$ which is also the projection of the intersection of the hyperplanes

$$\mathcal{D}(\lambda_d(\mathbf{p})) = \{\hat{\mathbf{x}} \in \mathbb{R}^{d+1} : x_d = 2\mathbf{p}\cdot\mathbf{x} - ||\mathbf{p}||^2\} \quad \text{and} \quad \mathcal{D}(\lambda_d(\mathbf{q})) = \{\hat{\mathbf{x}} \in \mathbb{R}^{d+1} : x_d = 2\mathbf{q}\cdot\mathbf{x} - ||\mathbf{q}||^2\}.$$

These are the duals of the lifted points which as seen are the tangent hyperplanes to the paraboloid at these points. Note that $\mathbf{x}$ is closer to $\mathbf{p}$ than to $\mathbf{q}$ iff $\pi_d^{-1}(\mathbf{x})$ intersects $\mathcal{D}(\lambda_d(\mathbf{p}))$ higher than $\mathcal{D}(\lambda_d(\mathbf{q}))$.

Delaunay/Voronoi as Polyhedra Projections. The Delaunay and Voronoi complexes in $\mathbb{R}^d$ can be obtained as the projections of boundary complexes of polyhedra in $\mathbb{R}^{d+1}$. For $S \subseteq \mathsf{P}$, let

$$\hat{S} = \lambda_d(S) = \{(\mathbf{p}, ||\mathbf{p}||^2) : \mathbf{p} \in S\} \subseteq \hat{\mathsf{P}}$$

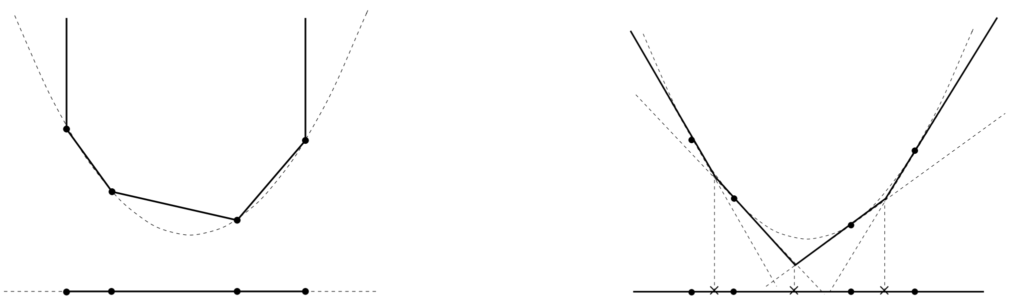

Figure 1: Delaunay and Voronoi diagrams via lifting and projection

◀ *(i)* $\mathsf{Del}(S) = \pi_d(\mathsf{LH}(\hat{S}))$ *and (ii)* $\mathsf{Vor}(S) = \pi_d(\mathsf{UE}(\mathcal{D}(\hat{S})))$.

Proof. (i) For $T \subseteq S$ with $|T| = d + 1$, $\mathrm{conv}(T) \in \mathsf{Del}(S)$ iff the interior of sphere(T) has no sites iff affine$(\mathrm{conv}(\hat{T}))$ has no lifted sites below iff $\mathrm{conv}(\hat{T})$ is in $\mathsf{LH}(\hat{S})$.
(ii) $\mathbf{x} \in V_{\mathbf{p}}$ iff for any other $\mathbf{q} \in S$, $||\mathbf{x} - \mathbf{q}||^2 \geq ||\mathbf{x} - \mathbf{p}||^2$ iff $\pi^{-1}(\mathbf{x})$ intersects $\mathcal{D}(\lambda(\mathbf{p}))$ higher than any other $\mathcal{D}(\lambda(\mathbf{q}))$ iff $\pi^{-1}(\mathbf{x})$ intersects the facet $\hat{V}_{\mathbf{p}}$ of $\mathsf{UE}(\mathcal{D}(\hat{S}))$. This follows from the previous argument on lifting bisectors. ☐